Influence of noise on the synchronization of the stochastic Kuramoto model

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We consider the Kuramoto model of globally coupled phase oscillators subject to Ornstein-Uhlenbeck and non-Gaussian colored noise and investigate the influence of noise on the order parameter of the synchronization process. We use numerical methods to study the dependence of the threshold as well as the maximum degree of synchronization on the correlation time and the strength of the noise, and find that the threshold of synchronization strongly depends on the nature of the noise. It is found to be lower for both the Ornstein-Uhlenbeck and non-Gaussian processes compared to the case of white noise. A finite correlation time also favors the achievement of the full synchronization of the system, in contract to the white noise process, which does not allow that. Finally, we discuss possible applications of the stochastic Kuramoto model to oscillations taking place in biochemical systems.

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I. INTRODUCTION

Synchronization plays a key role in many processes taking place in nature, laboratories, and theoretical model systems [1–10]. Systems that show synchronized behaviors include biological clocks [1], chemical oscillators [2], coupled phase oscillators [1–6], coupled respiratory and cardiac organs [7], and coupled map lattices [8–10]. In the present paper, we will study the synchronization behavior of globally coupled phase oscillators subject to Ornstein-Uhlenbeck (OU) [11,12] and non-Gaussian colored noise [13–15].

The model first introduced by Kuramoto [2] is one of the basic models that describes the synchronization process when initially independent oscillators begin to move coherently. It has been thoroughly studied and successfully applied in several systems which were modeled by an ensemble of coupled phase oscillators [6]. The Kuramoto model subject to a noise source and known as the *stochastic Kuramoto model* has also been subject to intensive investigations [4–6]. Most research was carried out for white noise. This Gaussian noise with zero correlation time shifts the threshold of synchronization to a higher value of the coupling constant. That behavior is rather expected as a noise usually increases the threshold of transition to an ordered phase. Also, white noise decreases the maximum degree of synchronization, thus not allowing the system to achieve full synchronization. We wonder how the Kuramoto model will behave in the case of other noise sources.

In this paper, we consider the Kuramoto model of globally coupled phase oscillators (each oscillator couples with all other oscillators) subject to OU noise, that is, a Gaussian process with finite correlation time, and non-Gaussian noise [13]. We are motivated by observations that the nature of noise in, e.g., living systems is not always thermal, that is, described by the Gaussian white noise process, but rather non-Gaussian [14].

The focus in our study is on the influence of noise on the synchronization process. We first review the known facts about the deterministic Kuramoto model as well as about the Kuramoto model with white noise. Then we consider the stochastic Kuramoto model subject to the OU noise process. The model of globally coupled phase oscillators with OU noise was previously considered in [16], where the effective frequency was obtained. Our goal is to study the dependence of the threshold and the degree of synchronization on the strength of the noise and the correlation time. Then we will turn to a non-Gaussian noise process and will find its effects on the synchronization of the phase oscillators. Finally, we outline a possible application of this and related models to oscillations taking place in biochemical systems, e.g., circadian clocks.

II. STOCHASTIC KURAMOTO MODEL

We begin with an introduction to the deterministic Kuramoto model [2], which describes N coupled phase oscillators with dynamics governed by the equations

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$$\frac{d\theta_i}{dt} = \omega_i + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where θ_i is the phase of the ith oscillator having frequency ω_i, and ϵ is the coupling constant.

The quantity of interest is

$$Z = \Gamma e^{i\Theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \tag{2}$$

which would be the order parameter to measure the extent of synchronization in the system of N phase oscillators. Its magnitude Γ determines the degree of synchronization. It can be seen that when all the oscillators have the same phase the quantity equals one ($\Gamma=1$), which corresponds to full synchronization. The degree of synchronization is equal to zero ($\Gamma=0$) when all the oscillators are independent and have different phases. Θ defines the average phase of the oscillators. Using Γ, we can rewrite Eq. (1) as

$$\frac{d\theta_i}{dt} = \omega_i + \epsilon\Gamma\sin(\Theta - \theta_i). \tag{3}$$

Equations (2) and (3) and (3) form a self-consistent set of equations to solve. For the initial distribution of frequencies we choose the Lorentzian distribution

$$g(w) = \frac{1}{\pi}\frac{\lambda}{(\omega - \bar{\omega})^2 + \lambda^2}. \tag{4}$$

In the following we will take the average frequency as zero, $\bar{\omega}=0$. Having the above distribution for the frequencies helps to find from Eqs. (2) and (3) an analytical expression for the stationary value of the synchronization degree [4],

$$\Gamma = \sqrt{1 - \frac{2\lambda}{\epsilon}}. \tag{5}$$

The transition to synchronization from the incoherent state with $\Gamma=0$ occurs at the critical value $\epsilon_c=2\lambda$, and the degree of synchronization asymptotically reaches its maximal value $\Gamma_m=1$, the full synchronization of the ensemble of phase oscillators.

Let us now consider the stochastic Kuramoto model driven by noise $\eta_i(t)$,

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \eta_i(t), \tag{6}$$

where the independent noise processes are governed by [13,15]

$$\frac{d\eta_i}{dt} = -\frac{1}{\tau}\frac{d}{d\eta_i}U_p(\eta_i) + \frac{\sqrt{D}}{\tau}\xi_i(t). \tag{7}$$

The potential function is

$$U_p(\eta) = \left(\frac{D}{\tau}(p-1)\right)\ln[1 + \alpha(p-1)\eta^2/2]$$

with $\alpha=\tau/D$. $\xi(t)$ is the Gaussian white noise process defined via $\langle\xi(t)\xi(t')\rangle=2\delta(t-t')$ and $\langle\xi(t)\rangle=0$. D and τ define the intensity and the correlation time of the noise process. The form of the noise η allows us to control the deviation from the Gaussian behavior by changing a single parameter p. For $p=1$, Eq. (7) becomes

$$\frac{d\eta_i}{dt} = -\frac{\eta_i}{\tau} + \frac{\sqrt{D}}{\tau}\xi_i(t), \tag{8}$$

which is a well-known time evolution equation for the OU noise process [11–13,15].

The correlation function of the OU noise [11,12] is given by

$$\langle\eta(t)\eta(0)\rangle = \frac{D}{\tau}e^{-t/\tau}. \tag{9}$$

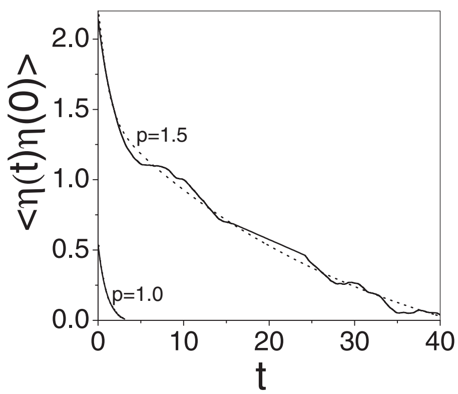

FIG. 1. Two-time correlation function vs t of both Gaussian and non-Gaussian noises for parameters $\tau=1.0$ and $D=0.5$.

Thus τ is the correlation time of the OU noise. To present the properties of the non-Gaussian noise we have plotted the two-time correlation function vs time in Fig. 1 via a numerical simulation. The curve for non-Gaussian noise ($p>1$) is fitted well by a biexponential decaying function (dotted curve) with correlation times $\tau=31$ and 1, respectively, for $p=1.5$. Figure 1 shows that the effective correlation time and the noise strength for $p>1$ are larger than those for $p=1.0$. Stationary properties of the noise η for $p>1$, including the time correlation function, have been studied in [17] and here we summarize the main results. The stationary probability distribution of η is

$$P(\eta) = \frac{1}{Z_p}\left(1 + \alpha(p-1)\frac{\eta^2}{2}\right)^{-1/(p-1)}, \tag{10}$$

where Z_p is the normalization factor given by

$$Z_p = \sqrt{\frac{\pi}{\alpha(p-1)}}\frac{\Gamma_1(1/(p-1)-1/2)}{\Gamma_1(1/(p-1))} \tag{11}$$

with Γ_1 being the Gamma function. This distribution can be normalized only for $p<3$. Since $P(\eta)$ is an even function of η, the first moment $\langle\eta\rangle$ is always equal to zero, and the second moment, given by

$$\langle \eta_p^2 \rangle = \frac{2D}{\tau(5-3p)}, \tag{12}$$

is well defined only for $p<5/3$. This shows that for a given external noise strength D and noise correlation time τ the variance of the non-Gaussian is higher than that of the Gaussian noise for $p>1$, as we have numerically shown above.

Now we check whether the above distribution function reduces to the Gaussian form for $p=1$. In this limit the term in large parentheses of Eq. (10) can be written as $1+\alpha(p-1)\eta^2/2=\exp[\alpha(p-1)\eta^2/2]$ and therefore Eq. (10) becomes

$$P(\eta) = \frac{1}{Z_1}\exp(-\alpha\eta^2/2), \tag{13}$$

with $Z_1=\sqrt{\pi/\alpha}$. Thus it is the stationary distribution function of colored Gaussian noise whose time evolution is given by Eq. (9). Finally, for $p<1$, the distribution function for η has a cutoff and it is defined only for $|\eta|<\eta_c\equiv\sqrt{2D/\tau(1-p)}$. As the distribution function is not defined for arbitrary values of η, we are not interested in exploring the limit $p<1$ in the present paper.

The case of white noise corresponds to $p=1$ and $\tau=0$ (δ-correlated Gaussian process). For this case it can be shown, beginning with the Fokker-Planck equation for the phase variables θ_i, that one can derive an equation for the order parameter Γ [18] from which the critical value of the interaction parameter ϵ_c can be obtained. For the general case of an arbitrary distribution of frequencies $g(\omega)$, the threshold value in the limit $N\to\infty$ can be obtained via

$$\epsilon_c = \frac{2}{\int d\omega\, g(\omega) D/(D^2+\omega^2)}, \tag{14}$$

which becomes $\epsilon_c=2/\pi g(0)$ when noise is absent ($D=0$). For the Lorentzian distribution of frequencies (4) the values of the critical coupling are $\epsilon_c=2\lambda$ for the deterministic and $\epsilon_c=2(\lambda+D)$ for the white noise cases [18].

In order to understand the thermodynamic limit in terms of numbers of coupled phase oscillators, we have plotted the critical coupling strength vs the number of oscillators in Fig. 2. It shows that the system with $N=5000$ oscillators has the same behavior as the thermodynamic system. Thus we performed our simulations with $N=5000$, which gives results as in the thermodynamic limit. More evidence for this statement will be presented below.

III. RESULTS

It is difficult to deal with Eqs. (6) and (7) analytically as they involve both nonlinearity and finite-correlation-time noise. Instead we have solved the stochastic differential equations (6) and (7) simultaneously using Heun's method, a stochastic version of the Euler method which reduces to the second-order Runge-Kutta method in the absence of noise [19].

In Fig. 3 we have plotted the average value of the degree of synchronization $\langle\Gamma\rangle$ versus the coupling strength ϵ for the

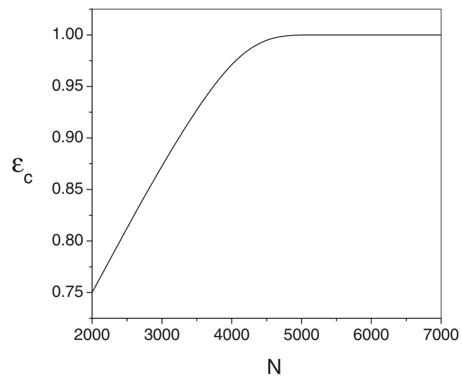

FIG. 2. Critical value of the coupling strength ϵ_c vs the number of coupled oscillators for $\lambda=0.5$ and $D=0.0$.

cases of no noise ($D=0$), white noise ($D=5.0$, $\tau=0.0$), and colored Gaussian noise ($D=5.0$, $\tau=0.5$) as solid, dashed, and dotted lines, respectively. In the presence of noise it is difficult to make the system synchronized because of the phase diffusion. As a result of that, the synchronization phase transition appears at a relatively higher value of coupling strength in the presence of noise (dotted and dashed curves) compared to the deterministic case (solid curve). Not only is the critical coupling strength affected, but also the maximum degree of synchronization is reduced in the presence of noise. The influence of white noise does not allow for the full synchronization of oscillators (dashed curve). However, the dotted curve shows that the noise correlation plays a constructive role in the synchronization phenomenon. Even at very small but finite correlation times, the degree of synchronization reaches a value of full synchronization similar to that in the deterministic case, demonstrating a low coupling

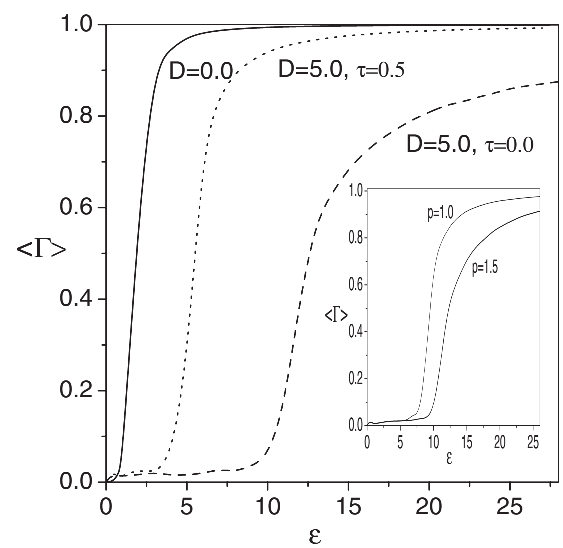

FIG. 3. Average magnitude of the order parameter $\langle\Gamma\rangle$ vs the coupling strength ϵ both in the presence (dashed and dotted curves) and in the absence (solid curves) of noise. The width of the Lorentzian distribution for initial frequencies is $\lambda=0.5$. In the inset, same plot for colored Gaussian and non-Gaussian noise, respectively. The parameter set for the inset is $\tau=0.5$, $D=15.0$, and $\lambda=0.5$.

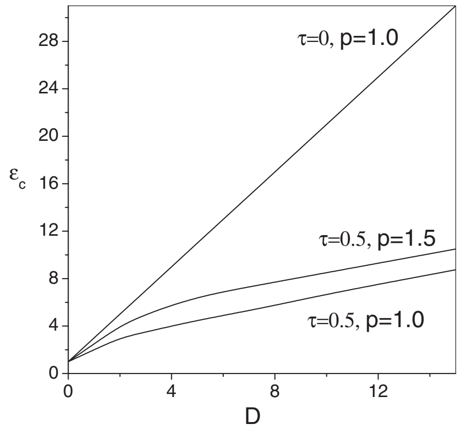

FIG. 4. Critical value of the coupling strength ϵ_c vs the noise strength D for white ($\tau=0$, $p=1$), OU ($\tau=0.5$, $p=1$), and non-Gaussian ($\tau=0.5$, $p=1.5$) noise sources. $\lambda=0.5$.

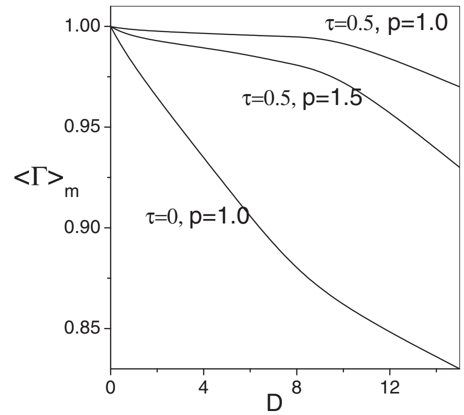

FIG. 5. Maximum value of the degree of synchronization $\langle\Gamma\rangle_m$ vs the noise strength D for white ($\tau=0$, $p=1$), OU ($\tau=0.5$, $p=1$), and non-Gaussian ($\tau=0.5$, $p=1.5$) noise sources with $\lambda=0.5$ for all cases.

strength and high degree of synchronization. This can be explained in the following way. The time correlations of the colored noise develop intrinsic correlations among the phase oscillators that lead to a high degree of synchronization at small coupling strengths.

Now we try to find out how these quantities are affected if the colored noise becomes non-Gaussian. We have plotted $\langle\Gamma\rangle$ versus ϵ in the inset of Fig. 3 for both colored Gaussian and non-Gaussian noise. The curves show that the synchronization for the non-Gaussian noise occurs at a higher threshold compared to that for Gaussian OU noise. This happens because of the higher value of the second moment [see Eq. (12) and Fig. 1] of the non-Gaussian noise than that of the Gaussian noise.

In the next step, we compare the variation of critical coupling strength ϵ_c with the noise strength D for colored noise to the known linear relation $\epsilon_c=2(\lambda+D)$ for white noise, and plot ϵ_c versus the noise strength D in Fig. 4; ϵ_c is determined numerically from the plot of $\langle\Gamma\rangle$ vs ϵ as in Fig. 3. The value of the ϵ at which the phase transition takes place corresponds to the value of the critical coupling strength ϵ_c. The linear plot in Fig. 4 for white noise follows the relation $\epsilon_c=2(\lambda+D)$ for a thermodynamic system with $N\to\infty$ and implies that our present numerical calculations at $N=5000$ represent very well the results in the thermodynamic limit $N\to\infty$. However, Fig. 4 shows that for both colored Gaussian and non-Gaussian noise the critical coupling ϵ_c increases very slowly compared to the case of white noise. This is a signature of intrinsic correlations developed among the phases due to the finite correlation time of the noise.

The same mechanism explains the decay of the average maximum degree of synchronization $\langle\Gamma\rangle_m$ with the increase of noise strength D as plotted in Fig. 5. The figure shows that the decay rate of the white noise ($\tau=0$, $p=1.0$) is larger than those of the colored Gaussian ($\tau=0.5$, $p=1.0$) and colored non-Gaussian ($\tau=0.5$, $p=1.5$) noise. For the colored non-Gaussian noise the decay rate is higher compared to the colored Gaussian noise as the variance is higher for the former case than for the latter.

To understand the dependence of the critical coupling strength ϵ_c and the average maximum degree of synchronization $\langle\Gamma\rangle_m$ on the correlation time τ of colored noises, we compute the quantities and present them in Fig. 6 and its inset, respectively. The figure shows that for both the Gaussian and non-Gaussian noise the critical coupling parameter ϵ_c decays biexponentially (the dotted curves are due to biexponentially decaying fitting) with increase of the noise correlation time τ. It differs from Fig. 1 in Ref. [20], where the variance was kept fixed and the critical coupling strength decreases due to increase of damping in the time evolution equation of the noise. But in Fig. 6 the variance decreases with increase of noise correlation time. Therefore, as the correlation time of the noise increases, the phase diffusion is reduced and intrinsic correlations among the phases develop strongly. As a result, the value of the synchronization threshold ϵ_c decreases with increase of the correlation time τ. The rate of decrease is very high at small τ's, since in this limit

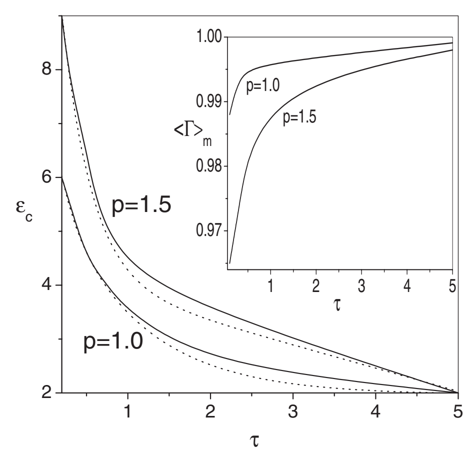

FIG. 6. Critical value of coupling strength ϵ_c vs the noise correlation time τ for OU ($p=1$) and non-Gaussian ($p=1.5$) noise. The other parameters are $\lambda=0.5$ and $D=5.0$. In the inset the maximum value of the degree of synchronization $\langle\Gamma\rangle_m$ vs the correlation time τ is presented for OU ($p=1$) and non-Gaussian ($p=1.5$) noise. The parameters are the same as in the main figure. The dotted curves show a biexponential fit.

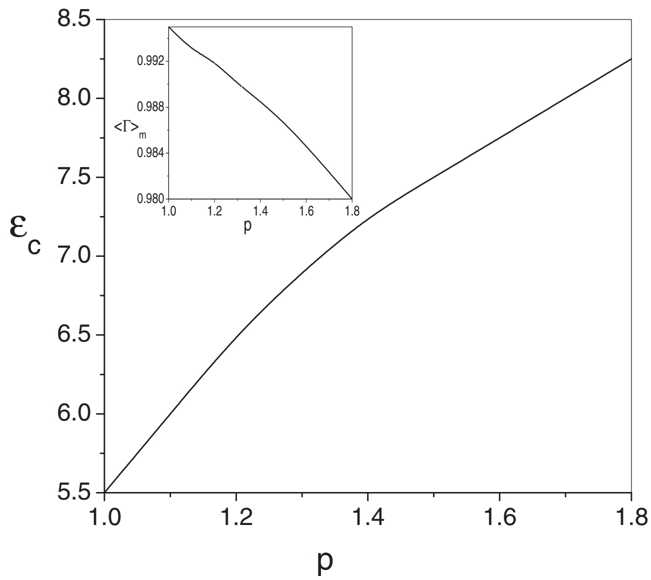

FIG. 7. Critical value of the coupling strength ϵ_c vs the parameter p that characterizes the deviation from Gaussian behavior. The parameters are $\lambda=0.5$, $\tau=0.5$, and $D=7.5$. In the inset the maximum degree of synchronization $\langle\Gamma\rangle_m$ is plotted vs p for the same set of parameters as in the main figure.

the noise strength has the dominating role in the dynamics and the correlation effect tries to overcome it. However, at small correlation times the large value of the critical coupling strength for the non-Gaussian noise compared to the Gaussian case is due to the very large value of its second moment [see Eq. (12) and Fig. 1] compared to the Gaussian noise, and thus the noise correlations have little effect on the synchronization. But at large values of τ the effective correlations of the non-Gaussian noise become very high (see Fig. 1) compared to those of the Gaussian noise, such that the effect of the excess noise strength is nullified and the critical coupling parameter ϵ_c becomes the same for both types of noise. The above explanations suggest that the degree of maximum synchronization should increase as the noise correlation time grows. One can also expect that the rate of increase is higher at small correlation times compared to large values of τ. The results of our numerical simulations agree with this, as presented in the inset of Fig. 6.

Finally, we demonstrate the variation of the synchronization threshold ϵ_c with the parameter p in Fig. 7. It shows that when p is close to 1, which corresponds to Gaussian processes, then the critical coupling strength increases at a higher rate than at large values of p. This result can be understood in the following way. As the noise deviates from Gaussian behavior, the effective noise strength and correlation time increase and in the limit of $p\to 1$ the former dominates over the latter. Thus the rate of increase of phase diffusion decreases with increase of p in the interplay of noise strength and the correlation time. Meanwhile, the critical coupling ϵ_c grows at a slower rate as the noise deviates more from Gaussian. However, the maximum degree of synchronization decreases more or less at the same rate as p grows. It is displayed in the inset of Fig. 7.

IV. SUMMARY AND DISCUSSION

In summary, we have considered the stochastic Kuramoto model of globally coupled phase oscillators subject to both Gaussian and non-Gaussian noise. The main focus was on the influence of noise on the synchronization phenomenon. The dependence of the critical value of the coupling strength ϵ_c on the strength of noise D and correlation time τ was thoroughly studied. A remarkable result is the decrease of ϵ_c in the case of the OU process compared to the case of the white noise process. That occurs due to the finite-time correlations of the noise, which develop intrinsic correlations between the phases. Also, the threshold tends to decrease with increase of the correlation time. The comparison between the Gaussian OU process and a non-Gaussian process (which reduces to the OU noise for a particular value of a parameter that controls the deviation from the Gaussian behavior) showed that the OU noise source advances the synchronization compared to the non-Gaussian noise. This is caused by the fact that the variance is higher for the non-Gaussian case [Eq. (12)].

We have also investigated the dependence of the maximal degree of synchronization $\langle\Gamma\rangle_m$ on the characteristics of the noise sources. As expected, it does decrease with increase of the noise strength D. However, we have obtained the result that both the OU and the non-Gaussian types of noise do allow the system to reach full synchronization, in contrast to the case of white noise. We have also revealed that higher correlation times favor higher degrees of maximal synchronization. This can be explained in the same manner as the influence of the correlation time of the noise process.

It should be pointed out that many processes taking place in nature may involve noise with non-Gaussian properties, as was shown in [14]. That is why investigation of models that describe such systems with non-Gaussian noise should attract more attention. In our present attempt, we compared the impact of a non-Gaussian noise on the nonlinear behavior of a system of coupled phase oscillators with the influence of white noise, and came to the conclusion that observations of those deviations from the behavior predicted by white noise may serve as a signature for the nature of the noise source.

We have also found that the case of the OU process, that is a Gaussian noise with finite correlation time, qualitatively differs from the case of white noise. It is obviously more realistic for the description of real complex systems and appears to favor critical phenomena such as the synchronization process. We came to this conclusion due to the following facts. Complex systems in the real world are associated with a thermal environment (TE) as well as a nonthermal environment (NTE). By the TE, we mean the medium in which system is immersed. If the TE is continuous then there is a cutoff in the frequency of the bath vibrational modes and it leads to (colored) noise of finite correlation time [21]. Another strong origin of the color noise is the NTE as a result of complex nonlinear dynamics in the environmental degrees of freedom. Because of nonlinear dynamics, the noise from the NTE might be non-Gaussian in character. Our present study shows that experimentalists can prove the presence of colored noise with finite correlation time if measurements show that, e.g., a system achieves full synchronization, which is not possible for a system with white noise.

Very recently, the collective synchronization in a population of coupled identical phase oscillators with drifting frequencies was studied in the biological context in [20], motivated by recent experiments [22]. The drift of the frequencies was modeled as a process subject to OU noise. The model was applied to describe cell-autonomous and self-sustained molecular oscillators, which drive circadian behavior and physiology in mammals [23]. Their model was shown to combine essential aspects of circadian clocks such as the stability of the limit cycle, fluctuations, and intercellular coupling. The stochastic Kuramoto model considered in our paper can be readily extended to describe circadian clocks under the influence of non-Gaussian noise.

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